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Application of Change of Basis in the Simplex Method

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Abstract

The simplex method is a very useful method to solve linear programming problems. It gives us a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function. It is executed by performing elementary row operations on a matrix that we call the simplex tableau. It is an iterative method that by repeated use gives us the solution to any n variable linear programming model. In this paper, we apply the change of basis to construct following simplex tableaus without applying elementary row operations on the initial simplex tableau.

Keywords: change of basis, linear programming, simplex method, optimization, linear algebra

Introduction

In the summer of 1947, George B. Dantzig started to work on the simplex method for solving linear programs. The linear programming problem is to find

$$\min z,\ \boldsymbol{x} \geq 0 \text{ such that } A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{cx} = z_{min}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, A is an m by n matrix, and b and c are column and row vectors [1].

He presented in his work titled "Maximization of a linear function of variables subject to linear inequalities" the details of the simplex method by means of linear algebra [2]. The significance of this work lies in showing that we can do something about finding an optimal solution if such one exists. This method allows us to compute the optimal solution. Two issues in the simplex method are of great importance: First, with the simplex method we can obtain a basic feasible solution with which to start the computation and second, the simplex method ensures that the algorithm finishes in a finite number of steps either with an optimal solution or with the conclusion that there is no optimal solution [3].

Basis and coordinates[4]

Let V be a vector space and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be a set of vectors in V. B forms a basis for V if the following two conditions hold:

1. B is linearly independent.
2. B spans V.

If $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is a basis for V, then every vector $\boldsymbol{x} \in V$ can be expressed uniquely as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$.

$$\boldsymbol{x} = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2 + \dots + y_n\boldsymbol{v}_n \tag{1}$$

Theorem 1.: If $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is a basis for a vector space V, then every vector $\boldsymbol{x}$ in V can be written in one and only one way as a linear combination of vectors in B.

Proof 1.: Suppose there are two sets of coefficients for $\boldsymbol{x}$.

$$\boldsymbol{x} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \dots + k_n\boldsymbol{v}_n \tag{2}$$

and also

$$\boldsymbol{x} = l_1\boldsymbol{v}_1 + l_2\boldsymbol{v}_2 + \dots + l_n\boldsymbol{v}_n \tag{3}$$

Subtracting the two expressions for x gives

$$\boldsymbol{0} = (k_1 - l_1)\boldsymbol{v}_1 + (k_2 - l_2)\boldsymbol{v}_2 + \dots + (k_n - l_n)\boldsymbol{v}_n \tag{4}$$

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is linearly independent, so the coefficients in this expression must vanish:

$(k_1 - l_1) = 0$ implies $k_1 = l_1$

$(k_2 - l_2) = 0$ implies $k_2 = l_2$

…

$(k_n - l_n) = 0$ implies $k_n = l_n$

(5)

Therefore, the coefficients $k_1, k_2, \dots, k_n$ are unique as claimed.

Definition 1.: The coordinates of a vector $\boldsymbol{x}$ in a vector space V with respect to a basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ are those coefficients (y_i) which uniquely express $\boldsymbol{x}$ as linear combination of the basis vectors.

$$\boldsymbol{x} = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2 + \dots + y_n\boldsymbol{v}_n = y_{1j}\boldsymbol{v}_1 + y_{2j}\boldsymbol{v}_2 + \dots + y_{nj}\boldsymbol{v}_n \tag{6}$$

These coefficients $y_1, y_2, \dots, y_n$ are called the coordinates of $\boldsymbol{x}$ relative to the basis ($y_i \in \mathbb{R}$). The coordinate matrix (or coordinate vector) of relative to B is the column matrix in $\mathbb{R}^n$ whose components are the coordinates of $\boldsymbol{x}$.

$$[\boldsymbol{x}]_B = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \dots \\ y_{nj} \end{bmatrix} \quad (7)$$

In Figure 1, two coordinate systems in the plane are displayed:

- coordinate plane xy
- coordinate plane $x'y'$

Every coordinate system is defined by a basis.

- The standard coordinate system is defined by the standard basis:

$$S = (\boldsymbol{e}_1, \boldsymbol{e}_2) = \{(1,0), (0,1)\} \quad (8)$$

- The dashed coordinate system (non-standard) is defined by the basis:

$$B = (\boldsymbol{u}_1, \boldsymbol{u}_2) = \{(3,2), (-2,1)\} \quad (9)$$

In Figure 2, the vector $\boldsymbol{u} = (1,3)$ has standard coordinates $x = 1$ and $y = 3$.

If we use the dashed coordinate system (non-standard), whose coordinate axes are labelled x' and y'; the dashed coordinates of $\boldsymbol{u}$ are $x' = 1$ and $y' = 1$.

3. Change of Basis[4]

If we are provided with the coordinate matrix of a vector relative to one basis B and are asked to find the coordinate matrix of the vector relative to another basis B', we have to apply the procedure of change of basis. This is shown in Example 1.

Example 1.: Find the coordinate matrix of $\boldsymbol{x} = \{1, -2, -1\}$ in $\mathbb{R}^3$ relative to non-standard basis $B' = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \{(0,0,-1), (1,3,-1), (2,1,1)\}$.

Solution 1.: First, $\boldsymbol{x}$ is written as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

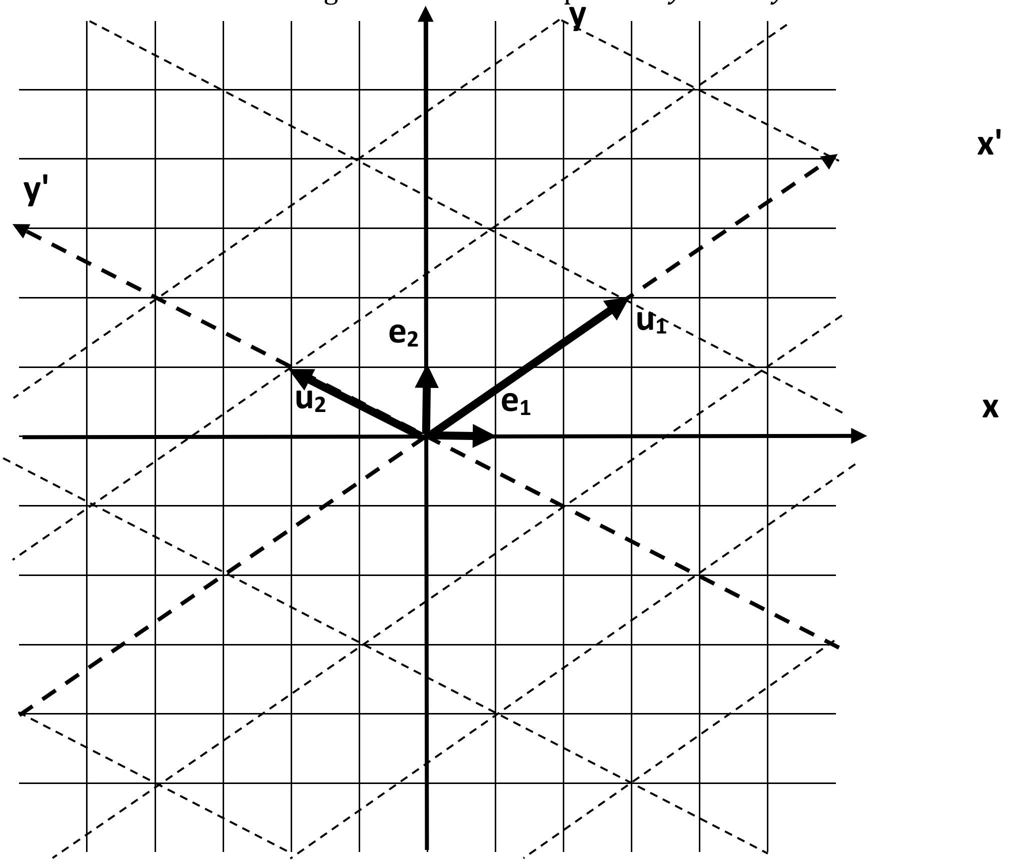
$$\boldsymbol{x} = y_1\boldsymbol{u}_1 + y_2\boldsymbol{u}_2 + y_3\boldsymbol{u}_3 \quad (10)$$

$$(1, -2, -1) = y_1(0,0,-1) + y_2(1,3,-1) + y_3(2,1,1) \quad (11)$$

Then, the following system of linear equations is obtained.

$$\begin{aligned} y_2 + 2y_3 &= 1 \\ 3y_2 + y_3 &= -2 \\ -y_1 - y_2 + y_3 &= -1 \end{aligned} \quad (12)$$

Figure 1: Coordinate planes xy and $x'y'$



This can be written in matrix form $P \cdot [\boldsymbol{x}]_{B'} = [\boldsymbol{x}]_B$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \qquad (13)$$

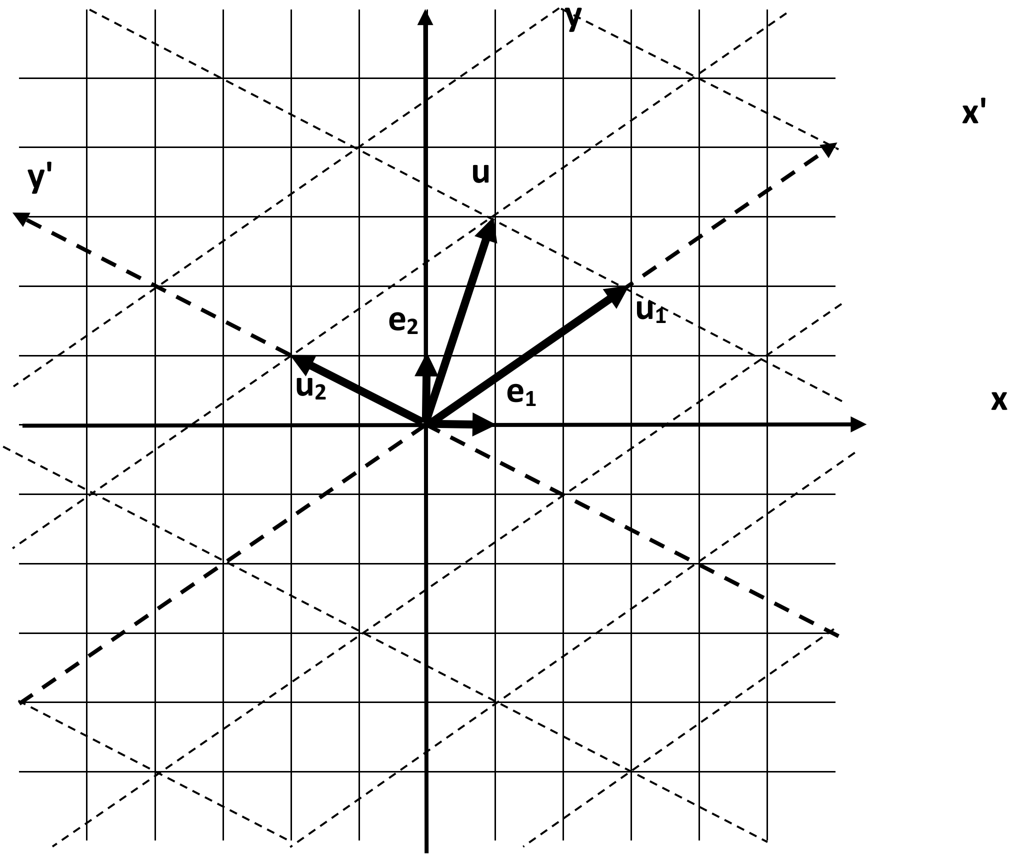
Where P is the transition matrix from B' to B, $[\boldsymbol{x}]_{B'}$ is the coordinate matrix of x relative to the basis B' and $[\boldsymbol{x}]_B$ is the coordinate matrix of $\boldsymbol{x}$ relative to the basis B. (13) shows the change of basis from B' to B.

$[\boldsymbol{x}]_{B'}$ can be found by $[\boldsymbol{x}]_{B'} = P^{-1} \cdot [\boldsymbol{x}]_B$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \qquad (14)$$

Where P^{-1} is the transition matrix from B to B'. So the solution of the system given in (12) is $y_1 = 3$, $y_2 = -1$ and $y_3 = 1$, so the coordinate matrix of $\boldsymbol{x}$ relative to B' is

Figure 2: Vector $\boldsymbol{u}$ in both coordinate systems



$$[\boldsymbol{x}]_{B'} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \tag{15}$$

Theorem 2.: Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ and $B' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ be two ordered bases for $\mathbb{R}^n$. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the matrix $[B'\colon B]$ as follows:

$$[B'\colon B] \longrightarrow [I_n\colon P^{-1}] \tag{16}$$

Example 2. shows an application of (16).

Example 2.: Find the transition matrix from B to B' for the following bases in $\mathbb{R}^3$.

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B' = \{(1,-1,0), (-2,1,2), (1,-1,-1)\}$

Solution 2.: First, B and B' are written in matrix form.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix} \quad (17)$$

The matrix $[B'\colon B]$ is formed and Gauss-Jordan Elimination is used to rewrite $[B'\colon B]$ as $[I_n\colon P^{-1}]$.

$$[B'\colon B] = \left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \dots$$

$$[I_n\colon P^{-1}] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right] \quad (18)$$

Transition matrix from B to B' is then

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ -2 & -2 & -1 \end{bmatrix} \quad (19)$$

Application

The simplex method is a very useful method to solve linear programming problems. It gives us a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function. It is executed by performing elementary row operations on a matrix that we call the simplex tableau. This tableau consists of augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$c_1x_1 + c_2x_2 + \ \dots + c_nx_n + 0 \cdot s_1 + 0 \cdot s_2 + \ \dots + 0 \cdot s_m - z = 0 \quad (20)$$

In this paper, we apply the change of basis to construct following simplex tableaus without applying elementary row operations on the initial simplex tableau.

Example 3.: $z_{max} = 2x_1 + x_2 + 3x_3$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 \quad\quad\ \leq 8 \\ & x_1 + \ x_2 + 2x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(21)

Solution 3.: $z_{max} = 2x_1 + x_2 + 3x_3 + 0 \cdot s_1 + 0 \cdot s_2$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 + 0 \cdot x_3 + 1 \cdot s_1 + 0 \cdot s_2 = 8 \\ & x_1 + \ x_2 + 2 \cdot x_3 + 0 \cdot s_1 + 1 \cdot s_2 = 12 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

(22)

Table 1: Initial simplex tableau

c_B	BASIC VARIABLES	c_j	2	1	3	0	0	
		x_B	x_1	x_2	x_3	s_1	s_2	
0	s_1	8	1	2	0	1	0	
0	s_2	12	1	1	2	0	1	MIN RATIO
z_j		0	0	0	0	0	0	
$z_j - c_j$		-----	-2	-1	-3	0	0	
					MIN			

Coefficient vectors of x_1, x_2, x_3, s_1, s_2 are respectively

$$\boldsymbol{a}_1 = \begin{bmatrix}1\\1\end{bmatrix} \ \boldsymbol{a}_2 = \begin{bmatrix}2\\1\end{bmatrix} \ \boldsymbol{a}_3 = \begin{bmatrix}0\\2\end{bmatrix} \ \boldsymbol{a}_4 = \begin{bmatrix}1\\0\end{bmatrix} \ \boldsymbol{a}_5 = \begin{bmatrix}0\\1\end{bmatrix} \quad (23)$$

In the initial simplex tableau in Table 1, the coefficient vectors that are in the basis B are

$$\boldsymbol{a}_4 = \begin{bmatrix}1\\0\end{bmatrix} \ \boldsymbol{a}_5 = \begin{bmatrix}0\\1\end{bmatrix} \quad (24)$$

$B = (\boldsymbol{a}_4, \boldsymbol{a}_5)$ (Basis of the initial simplex tableau)

In the initial simplex tableau, the pivot column is the coefficient vector of x_3, namely $\boldsymbol{a}_3$. The coefficients in the pivot column are the coordinates of $\boldsymbol{a}_3$ relative to the basis B.

After pivoting in the initial simplex tableau, we decided that s_2 is leaving the solution as x_3 is entering the solution. In the second simplex tableau, the coefficient vectors that are in the ordered basis B' are

$$\boldsymbol{a}_4 = \begin{bmatrix}1\\0\end{bmatrix} \ \boldsymbol{a}_3 = \begin{bmatrix}0\\2\end{bmatrix} \quad (25)$$

$B' = (\boldsymbol{a}_4, \boldsymbol{a}_3)$ (Basis of the second simplex tableau)

Without applying elementary row operations on the initial simplex tableau, we apply the change of basis to construct the second tableau. To get the transition matrix P^{-1}, the matrix $[B'\!:B]$ is formed and Gauss-Jordan Elimination is used to rewrite $[B'\!:B]$ as $[I_n\!:P^{-1}]$.

$$[B'\!:B] = \left[\begin{array}{cc:cc}1 & 0 & 1 & 0\\0 & 2 & 0 & 1\end{array}\right] \cdots$$

$$[I_n : P^{-1}] = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 0 & 1/2 \end{bmatrix} \quad (26)$$

Transition matrix from B to B' is then

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \quad (27)$$

To construct the second tableau in Table 2, we multiply the augmented matrix in the initial tableau with the transition matrix when the basis is changing from B to B'. So we get the augmented matrix of the second tableau

$$P^{-1} \cdot AUGMENTED\ MATRIX = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 8 & 1 & 2 & 0 & 1 & 0 \\ 12 & 1 & 1 & 2 & 0 & 1 \end{bmatrix} = \\ = \begin{bmatrix} 8 & 1 & 2 & 0 & 1 & 0 \\ 6 & 1/2 & 1/2 & 1 & 0 & 1/2 \end{bmatrix}$$

(28)

Table 2: Second simplex tableau

c_B	BASIC VARIABLES	c_j	2	1	3	0	0	
		x_B	x_1	x_2	x_3	s_1	s_2	
0	s_1	8	1	2	0	1	0	MIN RATIO
3	x_3	6	1/2	1/2	1	0	1/2	
z_j		18	3/2	3/2	3	0	3/2	
$z_j - c_j$		-----	-1/2	1/2	0	0	3/2	
			MIN					

Recall that in the initial simplex tableau the coefficients in the pivot column are the coordinates of $\boldsymbol{a}_3$ relative to the basis B. But for the second simplex tableau we have another basis B'. The coefficient vector of x_3 in the second simplex tableau gives us the coordinates of $\boldsymbol{a}_3$ relative to the basis B'.

First, $\boldsymbol{a}_3$ is written as a linear combination of $\boldsymbol{a}_4$ and $\boldsymbol{a}_3$.

$$(0,2) = c_1(1,0) + c_2(0,2) \quad (29)$$

Then, the following system of linear equations is obtained.

$$c_1 = 0$$

$$2c_2 = 2 \quad (30)$$

We can see that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (31)$$

So $[\boldsymbol{a}_3]_{B'}$ is

$$[\boldsymbol{a}_3]_{B'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{32}$$

The same holds for the coefficient vectors of x_1, x_2, s_1 and s_2 as well.

After pivoting in the second simplex tableau, we decided that s_1 is departing from the solution as x_1 is entering the solution. In the third simplex tableau, the coefficient vectors that are in the ordered basis B'' are

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{33}$$

$B'' = (\boldsymbol{a}_1, \boldsymbol{a}_3)$ (Basis of the third simplex tableau)

Without applying elementary row operations on the second simplex tableau, we apply the change of basis to construct the third tableau. To get the transition matrix $P'^{(-1)}$, the matrix $[B'' \colon B']$ is formed and Gauss-Jordan Elimination is used to rewrite $[B'' \colon B']$ as $\left[I_n \colon P'^{(-1)}\right]$.

$$[B'' \colon B'] = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 \\ 1 & 2 & \vdots & 0 & 2 \end{bmatrix} \dots$$

$$\left[I_n \colon P'^{(-1)}\right] = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & -1/2 & 1 \end{bmatrix} \tag{34}$$

Transition matrix from B' to B'' is then

$$P'^{(-1)} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \tag{35}$$

To construct the third tableau in Table 3, we multiply the augmented matrix in the second tableau with the transition matrix when the basis is changing from B' to B''. So we get the augmented matrix of the third tableau

$$P'^{(-1)} \cdot AUGMENTED\ MATRIX = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 8 & 1 & 2 & 0 & 1 & 0 \\ 6 & 1/2 & 1/2 & 1 & 0 & 1/2 \end{bmatrix} = \\ = \begin{bmatrix} 8 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & -1/2 & 1 & -1/2 & 1/2 \end{bmatrix}$$

(36)

In the third tableau there are no negative elements in the bottom row $z_j - c_j$. So the optimal solution is 22 monetary units (subsequently referred to as m.u.).

$$(x_1, x_2, x_3, s_1, s_2) = (8,0,2,0,0) \tag{37}$$

$$z_{max} = 2x_1 + x_2 + 3x_3 = 2 \cdot 8 + 1 \cdot 0 + 3 \cdot 2 = 22 \text{ m.u.} \tag{38}$$

Table 3: Third (optimal) simplex tableau

c_B	BASIC VARIABLES	c_j	2	1	3	0	0
		x_B	x_1	x_2	x_3	s_1	s_2
2	x_1	8	1	2	0	1	0
3	x_3	2	0	-1/2	1	-1/2	1/2
z_j		22	2	5/2	3	1/2	3/2
$z_j - c_j$		-----	0	3/2	0	1/2	3/2
			NO NEGATIVE ELEMENTS IN $z_j - c_j$				

Recall that in the initial simplex tableau the coefficients in the pivot column are the coordinates of $\boldsymbol{a}_3$ relative to the basis B. But for the third simplex tableau we have another basis B''. The coefficient vector of x_3 in the third simplex tableau gives us the coordinates of $\boldsymbol{a}_3$ relative to the basis B''.

First, $\boldsymbol{a}_3$ is written as a linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$.

$$(0,2) = c_1(1,1) + c_2(0,2) \quad (39)$$

Then, the following system of linear equations is obtained.

$$\begin{aligned} c_1 &= 0 \\ c_1 + 2c_2 &= 2 \end{aligned} \quad (40)$$

We can see that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (41)$$

So $[\boldsymbol{a}_3]_{B''}$ is

$$[\boldsymbol{a}_3]_{B''} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (42)$$

The same holds for the coefficient vectors of x_1, x_2, s_1 and s_2 as well.

1 Economic interpretation of the coordinate vectors in the optimal simplex tableau

The optimal simplex tableau in Table 3 shows that 8 units of x_1 and 2 units of x_3 should be produced to get 22 m.u. x_2 is a nonbasic variable which means that no unit of x_2 should be produced.

Let $\boldsymbol{y}_2$ be the coordinate vector of a_2 relative to the basis B'' in the optimal simplex tableau. So y_2 is

$$\boldsymbol{y}_2 = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix} = [\boldsymbol{a}_2]_{B''} \tag{43}$$

and we can obtain $\boldsymbol{a}_2$ by

$$B''\boldsymbol{y}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boldsymbol{a}_2 \tag{44}$$

$\boldsymbol{a}_2$ can be represented by a linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$.

$$\boldsymbol{a}_2 = 2\boldsymbol{a}_1 - \frac{1}{2}\boldsymbol{a}_3 \tag{45}$$

(45) tells us how much more or less we should produce of x_1 and x_3 if we want to produce one unit of x_2.

In the simplex algorithm basic variables can be represented by

$$\boldsymbol{x}_B = (B)^{-1}\boldsymbol{b} - \sum_{j\in J}[\boldsymbol{y}_j]x_j \tag{46}$$

where J is the set of the indices of the nonbasic variables [5]. Therefore, we can get

$$\frac{\partial x_B}{\partial x_j} = -\boldsymbol{y}_j \tag{47}$$

where $\left(-y_j\right)$ shows the rate of change of the basic variables as a function of the nonbasic variable x_j. If we increase x_j by one unit, the ith basic variable x_{Bi} should be decreased by an amount y_{ij}. This can be expressed by

$$\frac{\partial x_{Bi}}{\partial x_j} = -y_{ij} \tag{48}$$

Going back to Example 3, we have

$$\frac{\partial x_B}{\partial x_2} = -\boldsymbol{y}_2 = \begin{bmatrix} -y_{12} \\ -y_{22} \end{bmatrix} = \begin{bmatrix} -2 \\ 1/2 \end{bmatrix} \tag{49}$$

(49) tells us that if we want to produce one unit of x_2 we should decrease the production of x_1 by 2 units and increase the production of x_3 by ½ unit. Substituting these values for x_1, x_2 and x_3 in the constraints of Example 3, we see that

$$x_1 + 2x_2 \qquad \leq\ 8 \Rightarrow (8-2) + 2\cdot 1 = 8$$
$$x_1 + \ x_2 \ + 2x_3 \leq 12 \Rightarrow (8-2) + 1 + 2(2 + 1/2) = 12$$
(50)

are satisfied in equality. But if we substitute these values in the objective function we see that

$$z_{max} = 2x_1 + x_2 + 3x_3 = 2(8-2) + 1\cdot 1 + 3(2 + 1/2) = 41/2 \text{ m.u.} \tag{51}$$

gives us less profit than before. To make the same profit as before, we should increase the marginal profit of x_2. By using the trick $z_j = c_j + (z_j - c_j)$ we can calculate how much the new marginal profit of x_2 should be to make the same profit as before

$$z_2 = c_2 + (z_2 - c_2) = 1 + 3/2 = 5/2 \text{ m.u.} \quad (52)$$

(52) tells us that the marginal profit should be 5/2 m.u. to make the same profit as before because

$$z_{max} = 2x_1 + 5/2x_2 + 3x_3 = 2(8-2) + 5/2 \cdot 1 + 3(2 + 1/2) = 22 \text{ m.u.} \quad (53)$$

Conclusion

Every time the simplex algorithm calculates the next tableau, coefficient matrix of the original standard problem is multiplied by the inverse of the basis matrix of the actual tableau by using the formula $B^{-1}\boldsymbol{a}_j = \boldsymbol{y}_j$. This paper shows that the next tableau can be calculated by multiplying the transition matrix by the actual augmented matrix by using the formula $P^{-1}\boldsymbol{y}_j = \boldsymbol{y}'_j$.

In each tableau, the coordinate vector of a variable gives us the coordinates relative to the actual basis. In this paper, we made an economic interpretation of that coordinate vector.

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